

# On the Roman domination number of generalized Sierpiński graphs

F. Ramezani<sup>(1)</sup>, E. D. Rodríguez-Bazan<sup>(2)</sup>, J. A. Rodríguez-Velázquez<sup>(3)</sup>

<sup>(1)</sup>Department of Mathematics, Faculty of Science,

Imam Khomeini International University P. O. Box 34194-288, Qazvin, Iran.

ramezani@ikiu.ac.ir

<sup>(2)</sup>Department of Matemathics,

Central University of Las Villas, Carretera a Camajuaní km. 5½. Villa Clara, Cuba.

erickrodriguezbazan@gmail.com

<sup>(3)</sup>Departament d'Enginyeria Informàtica i Matemàtiques,

Universitat Rovira i Virgili, Av. Països Catalans 26, 43007 Tarragona, Spain.

juanalberto.rodriguez@urv.cat

May 24, 2016

## Abstract

A map  $f : V \rightarrow \{0, 1, 2\}$  is a Roman dominating function on a graph  $G = (V, E)$  if for every vertex  $v \in V$  with  $f(v) = 0$ , there exists a vertex  $u$ , adjacent to  $v$ , such that  $f(u) = 2$ . The weight of a Roman dominating function is given by  $f(V) = \sum_{u \in V} f(u)$ . The minimum weight of a Roman dominating function on  $G$  is called the Roman domination number of  $G$ . In this article we study the Roman domination number of Generalized Sierpiński graphs  $S(G, t)$ . More precisely, we obtain a general upper bound on the Roman domination number of  $S(G, t)$  and we discuss the tightness of this bound. In particular, we focus on the cases in which the base graph  $G$  is a path, a cycle, a complete graph or a graph having exactly one universal vertex.

*Keywords:* Roman domination number; Generalized Sierpiński graph; Sierpiński graph.

*AMS Subject Classification Numbers:* 05C69; 05C76.

## 1 Introduction

Let  $G = (V, E)$  be a non-empty graph of order  $n \geq 2$ , and  $t$  a positive integer. We denote by  $V^t$  the set of words of length  $t$  on alphabet  $V$ . The letters of a word  $u$  of

length  $t$  are denoted by  $u_1u_2\dots u_t$ . The concatenation of two words  $u$  and  $v$  is denoted by  $uv$ . Klavžar and Milutinović introduced in [12] the graph  $S(K_n, t)$ ,  $t \geq 1$ , whose vertex set is  $V^t$ , where  $\{u, v\}$  is an edge if and only if there exists  $i \in \{1, \dots, t\}$  such that:

- (i)  $u_j = v_j$ , if  $j < i$ ; (ii)  $u_i \neq v_i$ ; (iii)  $u_j = v_i$  and  $v_j = u_i$  if  $j > i$ .

As noted in [10], in a compact form, the edge sets can be described as

$$\{\{wu_iu_j^{r-1}, wu_ju_i^{r-1}\} : u_i, u_j \in V, i \neq j; r \in \{1, \dots, t\}; w \in V^{t-r}\}.$$

The graph  $S(K_3, t)$  is isomorphic to the graph of the Tower of Hanoi with  $t$  disks [12]. Later, those graphs have been called Sierpiński graphs in [13] and they were studied by now from numerous points of view. For instance, the authors of [6] studied identifying codes, locating-dominating codes, and total-dominating codes in Sierpiński graphs. In [9] the authors propose an algorithm, which makes use of three automata and the fact that there are at most two internally vertex-disjoint shortest paths between any two vertices, to determine all shortest paths in Sierpiński graphs. The authors of [13] proved that for any  $n \geq 1$  and  $t \geq 1$ , the Sierpiński graph  $S(K_n, t)$  has a unique 1-perfect code (or efficient dominating set) if  $t$  is even, and  $S(K_n, t)$  has exactly  $n$  1-perfect codes if  $t$  is odd. The Hamming dimension of a graph  $G$  was introduced in [14] as the largest dimension of a Hamming graph into which  $G$  embeds as an irredundant induced subgraph. That paper gives an upper bound for the Hamming dimension of the Sierpiński graphs  $S(K_n, t)$  for  $n \geq 3$ . It also showed that the Hamming dimension of  $S(K_n, t)$  grows as  $3^{t-3}$ . The idea of almost-extreme vertex of  $S(K_n, t)$  was introduced in [15] as a vertex that is either adjacent to an extreme vertex of  $S(K_n, t)$  or is incident to an edge between two subgraphs of  $S(K_n, t)$  isomorphic to  $S(K_n, t-1)$ . The authors of [15] deduced explicit formulas for the distance in  $S(K_n, t)$  between an arbitrary vertex and an almost-extreme vertex. Also they gave a formula of the metric dimension of a Sierpiński graph, which was independently obtained by Parreau in her Ph.D. thesis. For a general background on Sierpiński graph, the reader is invited to read the comprehensive survey [11] and references therein.

This construction was generalized in [7] for any graph  $G = (V, E)$ , by defining the  $t$ -th *generalized Sierpiński graph* of  $G$ , denoted by  $S(G, t)$ , as the graph with vertex set  $V^t$  and edge set defined as

$$\{\{wu_iu_j^{r-1}, wu_ju_i^{r-1}\} : \{u_i, u_j\} \in E; r \in \{1, \dots, t\}; w \in V^{t-r}\}.$$

Figure 1 shows a graph  $G$  and the generalized Sierpiński graph  $S(G, 2)$ , while Figure 2 shows the Sierpiński graph  $S(G, 3)$ .

Notice that if  $\{u, v\}$  is an edge of  $S(G, t)$ , there is an edge  $\{x, y\}$  of  $G$  and a word  $w$  such that  $u = wxyy\dots y$  and  $v = wyxx\dots x$ . In general,  $S(G, t)$  can be constructed recursively from  $G$  with the following process:  $S(G, 1) = G$  and, for  $t \geq 2$ , we copy  $n$  times  $S(G, t-1)$  and add the letter  $x$  at the beginning of each label of the vertices belonging to the copy of  $S(G, t-1)$  corresponding to  $x$ . Then for every edge  $\{x, y\}$  of  $G$ , add an edge between vertex  $xyy\dots y$  and vertex  $yxx\dots x$ . See, for instance, Figure

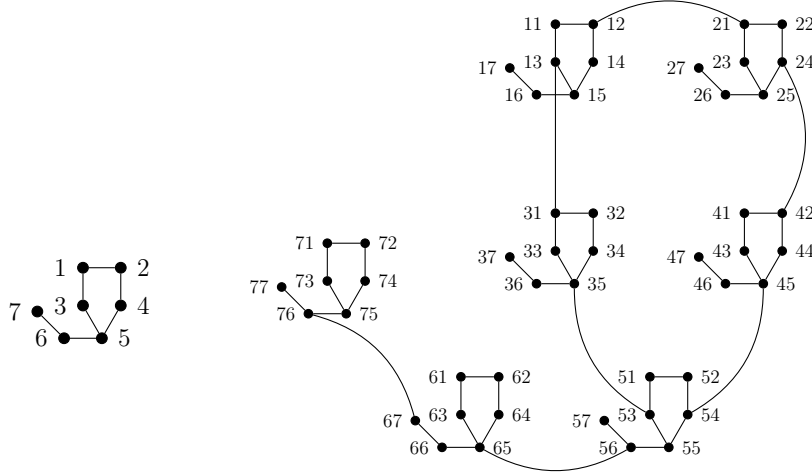


Figure 1: A graph  $G$  and the Sierpiński graph  $S(G, 2)$ .

2. Vertices of the form  $xx \dots x$  are called *extreme vertices* of  $S(G, t)$ . Notice that for any graph  $G$  of order  $n$  and any integer  $t \geq 2$ ,  $S(G, t)$  has  $n$  extreme vertices and, if  $x$  has degree  $d(x)$  in  $G$ , then the extreme vertex  $xx \dots x$  of  $S(G, t)$  also has degree  $d(x)$ . Moreover, the degrees of two vertices  $yx \dots x$  and  $xyy \dots y$ , which connect two copies of  $S(G, t-1)$ , are equal to  $d(x) + 1$  and  $d(y) + 1$ , respectively.

For any  $w \in V^{t-1}$  and  $t \geq 2$  the subgraph  $\langle V_w \rangle$  of  $S(G, t)$ , induced by  $V_w = \{wx : x \in V\}$ , is isomorphic to  $G$ . Notice that there exists only one vertex  $u \in V_w$  of the form  $w'xx \dots x$ , where  $w' \in V^r$  for some  $r \leq t-2$ . We will say that  $w'xx \dots x$  is the *extreme vertex* of  $\langle V_w \rangle$ , which is an extreme vertex in  $S(G, t)$  whenever  $r = 0$ . By definition of  $S(G, t)$  we deduce the following remark.

**Remark 1.** Let  $G = (V, E)$  be a graph, let  $t \geq 2$  be an integer and  $w \in V^{t-1}$ . If  $u \in V_w$  and  $v \in V^t - V_w$  are adjacent in  $S(G, t)$ , then either  $u$  is the extreme vertex of  $\langle V_w \rangle$  or  $u$  is adjacent to the extreme vertex of  $\langle V_w \rangle$ .

The authors of [7] announced some results about generalized Sierpiński graphs concerning their automorphism groups and perfect codes. These results definitely deserve to be published. Since then some papers have been published on various aspects of generalized Sierpiński graphs. For instance, in [17] their chromatic number, vertex cover number, clique number, and domination number, are investigated. The authors of [18] obtained closed formulae for the Randić index of polymeric networks modelled by generalized Sierpiński graphs, while in [4] this work was extended to the so-called generalized Randić index. Also, the total chromatic number of generalized Sierpiński graphs was studied in [5] and the strong metric dimension has recently been studied in [16]. In this paper we obtain closed formulae or bounds on the Roman domination number of generalized Sierpiński graphs  $S(G, t)$  in terms of parameters of the base graph  $G$ .

We begin by establishing the principal terminology and notation which we will use throughout the article. Hereafter  $G = (V, E)$  denotes a finite simple graph of order  $n \geq 2$ . The distance between two vertices  $x, y \in V$  will be denoted by  $d_G(x, y)$ .

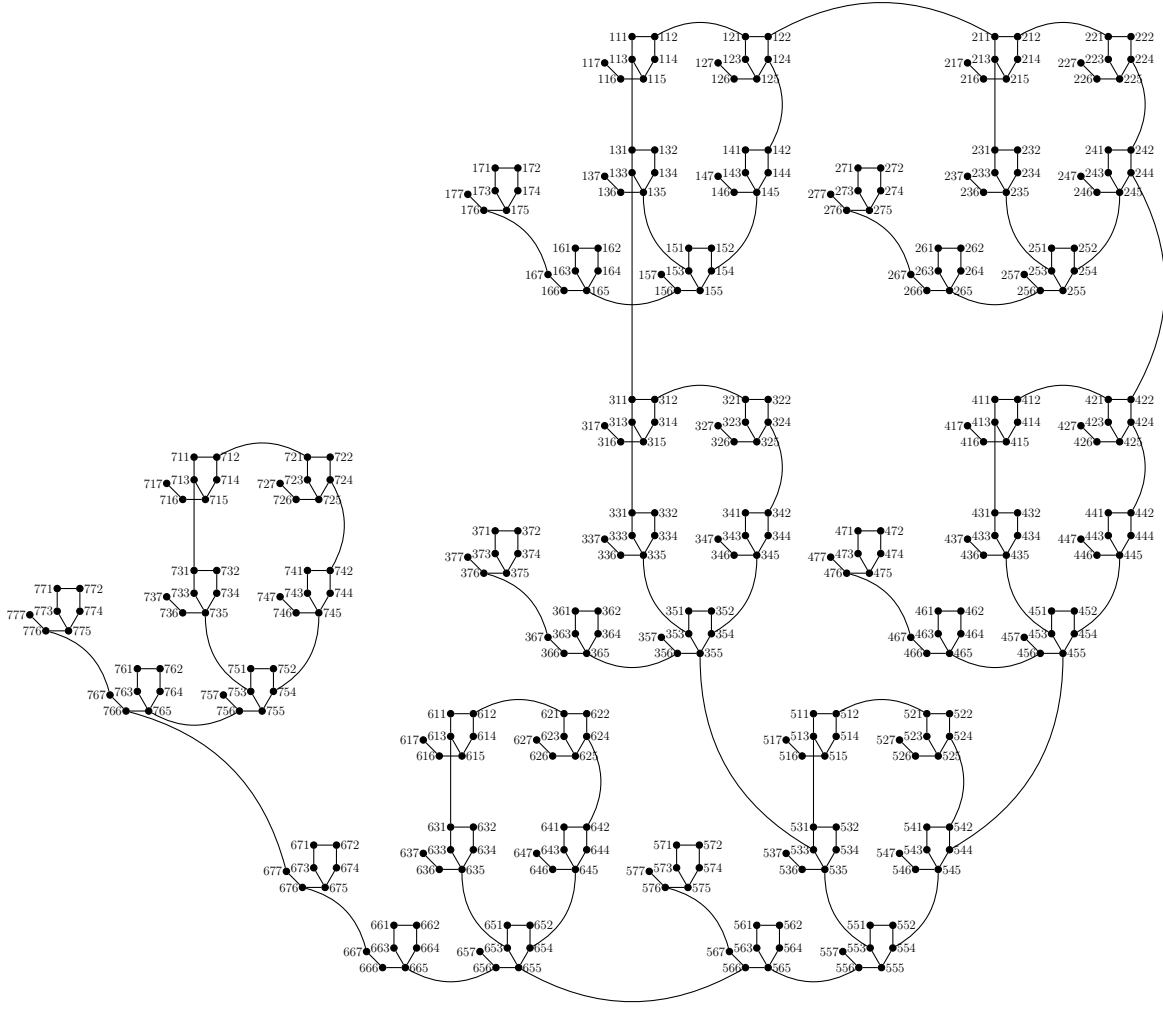


Figure 2: The Sierpiński graph  $S(G, 3)$  for the graph  $G$  of Figure 1.

For two adjacent vertices  $u$  and  $v$  of  $G$  we use the notation  $u \sim v$ . For a vertex  $v$  of  $G$ ,  $N(v) = \{u \in V : u \sim v\}$  denotes the set of neighbors that  $v$  has in  $G$ .  $N(v)$  is called the *open neighborhood* of  $v$  and the *close neighborhood* of  $v$  is defined as  $N[v] = N(v) \cup \{v\}$ . For a set  $D \subseteq V$ , the *open neighborhood* is  $N(D) = \cup_{v \in D} N(v)$  and the *closed neighborhood* is  $N[D] = N(D) \cup D$ . A set  $D$  is a *dominating set* if  $N[D] = V$ . The *domination number*  $\gamma(G)$  is the minimum cardinality of a dominating set in  $G$ . We say that a set  $S$  is a  $\gamma(G)$ -set if it is a dominating set and  $|S| = \gamma(G)$ . The subgraph induced a subset  $S$  of vertices will be denoted by  $\langle S \rangle$ .

A map  $f : V \rightarrow \{0, 1, 2\}$  is a *Roman dominating function* on a graph  $G$  if for every vertex  $v$  with  $f(v) = 0$ , there exists a vertex  $u \in N(v)$  such that  $f(u) = 2$ . The *weight* of a Roman dominating function is given by  $f(V) = \sum_{u \in V} f(u)$ . The minimum weight of a Roman dominating function on  $G$  is called the *Roman domination number* of  $G$  and it is denoted by  $\gamma_R(G)$ .

Any Roman dominating function  $f$  on a graph  $G$  induces three sets  $B_0, B_1, B_2$ , where  $B_i = \{v \in V : f(v) = i\}$ . Thus, we will write  $f = (B_0, B_1, B_2)$ . It is clear that

for any Roman dominating function  $f = (B_0, B_1, B_2)$  on a graph  $G = (V, E)$  of order  $n$  we have that  $f(V) = \sum_{u \in V} f(u) = 2|B_2| + |B_1|$  and  $|B_0| + |B_1| + |B_2| = n$ . We say that a function  $f = (B_0, B_1, B_2)$  is a  $\gamma_R(G)$ -function if it is a Roman dominating function and  $f(V) = \gamma_R(G)$ .

The Roman domination number was introduced by Cockayne et al. [3] in 2004 and since then about 100 papers have been published on various aspects of Roman domination in graphs (for examples, see [1, 2]). For instance, in [3, 8] was obtained the following result, which shows the relationship between the domination number and the Roman domination number of a graph.

**Lemma 2.** [3, 8] *For any graph  $G$ ,  $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$ .*

As shown in [3],  $\gamma(G) = \gamma_R(G)$  if and only if  $G$  is an empty graph. A graph  $G$  is said to be a *Roman graph* if  $\gamma_R(G) = 2\gamma(G)$ . Several examples of Roman graphs are given in [3, 19, 20].

**Theorem 3.** [3] *A graph  $G$  is Roman if and only if it has a  $\gamma_R$ -function  $f = (B_0, \emptyset, B_2)$ .*

The following result, stated in [3], will be used as a tool to study the Roman domination number of  $S(G, t)$  for the cases in which the base graph is a path or a cycle.

**Theorem 4.** [3] *For the classes of paths  $P_n$  and cycles  $C_n$ ,  $\gamma_R(P_n) = \gamma_R(C_n) = \lceil \frac{2n}{3} \rceil$ .*

Let  $G = (V, E)$  be a graph, and  $H = (V, E')$  a subgraph of  $G$ . Since any  $\gamma_R(H)$ -function is a Roman dominating function of  $G$ , we can state the following remark.

**Remark 5.** *Let  $G = (V, E)$  be a graph, and  $H = (V, E')$  a subgraph of  $G$ . Then  $\gamma_R(G) \leq \gamma_R(H)$ .*

## 2 An upper bound on the Roman domination number

Let  $f = (B_0, B_1, B_2)$  be a  $\gamma_R$ -function on  $G$  and let  $D_i$  be the set of non-isolated vertices of  $\langle B_i \rangle$  for  $i \in \{0, 1, 2\}$ . Also, let  $D_{12}$  be the set of non-isolated vertices of  $\langle B_1 \cup B_2 \rangle$ . Notice that, if we take  $f$  such that  $|B_1|$  is minimum, then  $B_1$  is an independent set, which implies that  $D_1 = \emptyset$  and  $D_{1,2} = D_2$ . With these notations in mind we state the following result.

**Theorem 6.** *Let  $G$  be a graph of order  $n$ . For any  $\gamma_R$ -function  $f = (B_0, B_1, B_2)$  on  $G$ , and any integer  $t \geq 2$ ,*

$$\gamma_R(S(G, t)) \leq n^{t-2}(n\gamma_R(G) - |B_2| - |D_{12}| - \theta + \frac{1}{2}|D_1|),$$

where  $\theta = |\{u \in B_1 \setminus D_1 : d_G(u, v) = 2 \text{ for some } v \in B_2 \text{ such that } |N_G(v) \cap B_0| = 2\}|$ .

*Proof.* Let  $f = (B_0, B_1, B_2)$  be a  $\gamma_R(G)$ -function. For a given integer  $t \geq 2$  we define  $S_i = \{wx; w \in V^{t-1}, x \in B_i\}$ , for  $i \in \{0, 1, 2\}$ . Let  $g : V^t \rightarrow \{0, 1, 2\}$  such that  $g = (S_0, S_1, S_2)$ . If  $v \in V^t$  and  $g(v) = 0$ , then  $v = wy$  where  $w$  is a word in  $V^{t-1}$  and  $y \in B_0$ . Since  $f$  is a  $\gamma_R$ -function on  $G$ , there is  $z \in B_2 \cap N_G(y)$ . Hence  $wz \in S_2 \cap N_{S(G,t)}(wy)$ . So  $g$  is a Roman dominating function on  $S(G, t)$  and  $\gamma_R(S(G, t)) \leq \omega(g) = n^{t-1}(|B_1| + 2|B_2|) = n^{t-1}\gamma_R(G)$ . Now we have four steps for reach the result.

Step 1: Set  $S'_2 = \{wuu : w \in V^{t-2}, u \in B_2\}$ . We define  $g_1 : V^t \rightarrow \{0, 1, 2\}$  such that  $g_1 = (S_0, S_1 \cup S'_2, S_2 \setminus S'_2)$ . Let  $y \in S_0$ . Then  $y$  has the form  $wuv_0$  where  $w \in V^{t-2}$ ,  $v_0 \in B_0$  and  $u \in V$ . Since  $f$  is a  $\gamma_R(G)$ -function, there is  $v_2 \in B_2$  such that  $v_0$  is adjacent to  $v_2$  in graph  $G$ . So  $wuv_0$  is adjacent to  $wuv_2$ . If  $wuv_2 \in S_2 \setminus S'_2$ , then we are done. Now, if  $wuv_2 \in S'_2$ , then  $v_2 = u$  and, since  $v_0$  is adjacent to  $v_2$ , we can conclude that  $y = wv_2v_0$  is adjacent to  $wv_0v_2$ . Hence  $g_1$  is a Roman dominating function on  $S(G, t)$ . Therefore  $\gamma_R(S(G, t)) \leq \omega(g_1) = n^{t-2}(n\gamma_R(G) - |B_2|)$ .

Step 2: Set  $S''_2 = \{wvv : w \in V^{t-2}, v \in D_2\}$ . We define  $g_2 : V^t \rightarrow \{0, 1, 2\}$  where

$$g_2(x) = \begin{cases} 0, & x \in S''_2; \\ g_1(x), & \text{otherwise.} \end{cases}$$

Let  $x \in V^t$  such that  $g_2(x) = 0$ . In this case,  $g_1(x) = 0$  or  $x \in S''_2$ .

Suppose that  $g_1(x) = 0$ . Since  $x$  must belong to  $S_0$ , it is of the form  $x = wuv_0$ , where  $w \in V^{t-2}$ ,  $u \in V$  and  $v_0 \in B_0$ . If  $N_{S(G,t)}(x) \cap S''_2 = \emptyset$ , then there exists  $y \in N_{S(G,t)}(x) \cap (S_2 \setminus S'_2)$ . On the other side, if  $z \in N_{S(G,t)}(x) \cap S''_2$ , then  $z = wv_2v_2$ , where  $v_2 \in D_2$  and  $u = v_2$ , and so  $v_2 \sim v_0$ , which implies that  $x = wv_2v_0 \sim wv_0v_2$ , and we know that  $g_2(wv_0v_2) = g_1(wv_0v_2) = g(wv_0v_2) = 2$ .

Now, if  $x \in S''_2$ , then there exists  $w \in V^{t-2}$  and  $v \in D_2$  such that  $x = wvv$ . So, by definition of  $D_2$ ,  $x$  must be adjacent to  $wvu$  for some  $u \in D_2 \setminus \{v\}$ . Hence  $g_2(wvu) = g_1(wvu) = g(wvu) = f(u) = 2$ .

Therefore,  $g_2$  is a Roman dominating function on  $S(G, t)$ , and so  $\gamma_R(S(G, t)) \leq n^{t-2}(n\gamma_R(G) - |B_2| - |D_2|)$ .

Step 3: We know that the maximum degree on  $\langle B_1 \rangle$  is one. Since  $D_1$  is the set of the non-isolated vertices of  $\langle B_1 \rangle$ ,  $\langle D_1 \rangle \cong \cup_{i=1}^k P_2$ , where  $k = \frac{1}{2}|D_1|$ . Suppose that  $\{v_1, u_1, v_2, u_2, \dots, v_k, u_k\}$  is the vertex set of  $\langle D_1 \rangle$ , where  $v_i \sim u_i$  for  $1 \leq i \leq k$ . Set  $S'_1 = \{wv_i v_i : w \in V^{t-2}, v_i \in D_1 \text{ and } 1 \leq i \leq k\}$ ,  $S''_1 = \{wu_i v_i : w \in V^{t-2}, v_i, u_i \in D_1 \text{ and } 1 \leq i \leq k\}$  and  $S'''_1 = \{wv_i u_i : wu_i v_i \in S''_1\}$ . We define  $g_3 : V^t \rightarrow \{0, 1, 2\}$  such that

$$g_3(x) = \begin{cases} 0, & x \in S'_1 \cup S''_1; \\ 2, & x \in S'''_1; \\ g_2(x), & \text{otherwise.} \end{cases}$$

Notice that  $S'''_1$  is a dominating set for  $S'_1 \cup S''_1$ . So  $g_3$  is a Roman dominating function on  $S(G, t)$ . Also  $\omega(g_3) = \omega(g_2) - |S'_1| - |S''_1| + |S'''_1|$  and  $|S'_1| = \frac{n^{t-2}}{2}|D_1|$ . Hence

$$\gamma_R(S(G, t)) \leq n^{t-2}(n\gamma_R(G) - |B_2| - |D_2| - \frac{1}{2}|D_1|).$$

We know that there are not any edges between  $B_1$  and  $B_2$ . So  $|D_{12}| = |D_1| + |D_2|$ . Hence  $\gamma_R(S(G, t)) \leq n^{t-2}(n\gamma_R(G) - |B_2| - |D_{12}| + \frac{1}{2}|D_1|)$ .

Step 4: Let  $B'_2 = \{v \in B_2 : |N_G(v) \cap B_0| = 2 \text{ and } d_G(v, u) = 2 \text{ for some } u \in B_1 \setminus D_1\}$ . Let  $\Pi$  be the set of paths  $v_0, w_2, w_0, w_1$  in  $G$  such that  $w_2 \in B'_2$ ,  $v_0, w_0 \in B_0$  and  $w_1 \in B_1 \setminus D_1$ . Given two vertices  $x, y \in V$ , we say that  $\mu(x, y) = (i, j)$  if there exist a path  $v_0, w_2, w_0, w_1$  in  $\Pi$  such that  $x$  and  $y$  are (from the left) in position  $i$  and  $j$ , respectively. We define the following sets.

$$A_1 = \{wxy : w \in V^{t-2} \text{ and } \mu(x, y) = (3, 4)\},$$

$$A_2 = \{wxy : w \in V^{t-2} \text{ and } \mu(x, y) = (4, 4)\},$$

$$A_3 = \{wxy : w \in V^{t-2} \text{ and } \mu(x, y) = (4, 2)\},$$

$$A_4 = \{wxy : w \in V^{t-2} \text{ and } \mu(x, y) = (4, 1)\},$$

$$A_5 = \{wxy : w \in V^{t-2} \text{ and } \mu(x, y) = (4, 3)\}.$$

Notice that  $|A_2| = \theta$  and  $A_i \cap A_j = \emptyset$ , for all  $i \neq j$ ,  $i, j \in \{1, \dots, 5\}$ . Also, since the weight of  $f$  is minimum, for every  $w_2 \in B'_2$  there exists exactly one vertex  $w_1 \in B_1 \setminus D_1$  such that  $d_G(w_2, w_1) = 2$ . Hence,  $|A_3| = |B'_2|$ . Furthermore, since  $|N_G(w_2) \cap B_0| = 2$ , we can conclude that  $|A_1| = |A_4| = |A_5|$ . On the other hand, suppose that there are two different paths  $v_0, w_2, w_0, w_1$  and  $v_0, w'_2, w'_0, w_1$  in  $\Pi$ . In such a case, the weight of the cycle  $v_0, w_2, w_0, w_1, w'_0, w'_2, v_0$  equals 5 and we can give a new  $\gamma_R(G)$ -function where the weight is 4, as we can consider that  $v_0$  and  $w_1$  have label 2 and the remaining vertices have label 0, which is a contradiction with the minimality of  $f$ . Hence,  $|A_4| = |B'_2|$ . Now, define the function  $g_4 : V^t \rightarrow \{0, 1, 2\}$  such that

$$g_4(v) = \begin{cases} 0, & v \in A_1 \cup A_2 \cup A_3; \\ 1, & v \in A_4; \\ 2, & v \in A_5; \\ g_3(v), & \text{otherwise.} \end{cases}$$

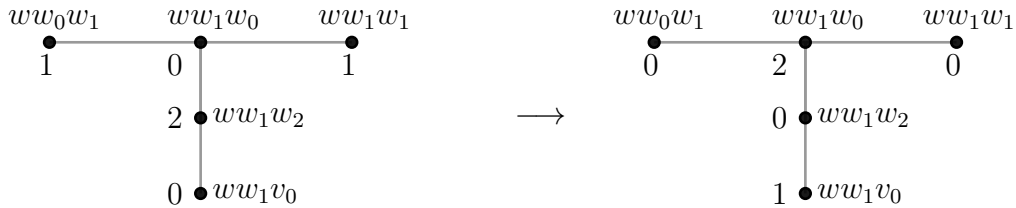


Figure 3: This figure shows how the labels imposed by function  $g_3$  are transformed by function  $g_4$ .

Notice that  $A_5$  is a dominating set for  $A_1 \cup A_2 \cup A_3$ . So  $g_4$  is a Roman dominating

function on  $S(G, t)$  (See Figure 3). Then

$$\begin{aligned}\omega(g_4) &= 2|A_5| + |A_4| + \omega(g_3) - |A_1| - |A_2| - 2|A_3| \\ &= \omega(g_3) - \theta \\ &\leq n^{t-2} \left( n\gamma_R(G) - |B_2| - |D_{12}| - \theta + \frac{|D_1|}{2} \right),\end{aligned}$$

as required.  $\square$

As we can see in Theorems 8 and 12 the bound above is achieved for any graph having exactly one universal vertex and for any path  $P_n$ , where  $n \equiv 0, 1 \pmod{3}$ .

Since any Roman graph has a  $\gamma_R$ -function  $f = (B_0, \emptyset, B_2)$ , we can state the following particular case of Theorem 6.

**Corollary 7.** *For any Roman graph  $G$  of order  $n$  and any integer  $t \geq 2$ ,*

$$\gamma_R(S(G, t)) \leq \gamma(G)n^{t-2}(2n-1).$$

### 3 Graphs having exactly one universal vertex

**Theorem 8.** *If  $G$  is a graph of order  $n \geq 4$  having exactly one vertex of degree  $n-1$ , then for any integer  $t \geq 2$ ,  $\gamma_R(S(G, t)) = n^{t-2}(2n-1)$ .*

*Proof.* By Theorem 6 we deduce that  $\gamma_R(S(G, t)) \leq n^{t-2}(2n-1)$ . We will show that for any  $\gamma_R$ -function  $f = (B_0, B_1, B_2)$  on  $S(G, t)$ ,  $\omega(f) \geq n^{t-2}(2n-1)$ . Let  $V = \{0, 1, \dots, n-1\}$  such that  $\deg(0) = n-1$ . We would point out that for any  $w \in V^{t-2}$ ,  $i \in V$  and  $t \geq 3$ , the subgraph  $\langle V_{wi} \rangle$  of  $S(G, t)$ , induced by  $V_{wi} = \{wij : j \in V\}$ , is isomorphic to  $G$ . Let  $\lambda(V_{wi}) = |\{wij \in V_{wi} : \deg(wij) \neq \deg(j)\}|$ . There are two general cases.

Case I.  $i \neq 0$ . In this case  $1 \leq \lambda(V_{wi}) \leq n-1$ . So there is  $wij \in V_{wi}$  such that  $\deg(wij) = \deg(j)$  for  $1 \leq j \leq n-1$ . If  $B_2 \cap V_{wi} \neq \emptyset$ , then  $\omega(V_{wi}) \geq 2$ . Otherwise,  $wij \in B_1$  and  $\omega(V_{wi}) \geq 1$ . If  $\omega(V_{wi}) = 1$ , then  $f(wik) = 0$  for  $k \in V \setminus \{j\}$ . Let  $l \in V \setminus \{0, i, j\}$ . Then  $wil \in N(wli)$  where  $wli \in B_2$ . Since  $l \neq 0$ ,  $\lambda(V_{wl}) \leq n-1$ , and so there is  $wll' \in V_{wl} \cap (B_1 \cup B_2)$  such that  $l' \neq i$ . Hence  $\omega(V_{wl}) \geq 3$ . This shows that  $\omega(V_{wi}) + \omega(V_{wl}) \geq 4$ . Therefore, for every copy of  $G$  of weight 1 there is another copy of  $G$  of weight at least 3. Since there are  $n^{t-2}(n-1)$  copies of  $G$  of this type in  $S(G, t)$ ,

the contribution of these copies of  $G$  to  $\omega(f)$  equals  $\sum_{w \in V^{t-2}} \sum_{i=1}^{n-1} \omega(V_{wi}) \geq 2n^{t-2}(n-1)$ .

Case II.  $i = 0$ . Then  $n-1 \leq \lambda(V_{w0}) \leq n$ . If  $V_{w0} \not\subseteq B_0$ , then  $\omega(V_{w0}) \geq 1$ . Suppose that  $\omega(V_{w0}) = 0$ . Hence  $\lambda(V_{w0}) = n$  and so  $w'00 \in N(w'jj)$  for  $w' \in V^{t-2}$  and  $j \neq 0$ . Since  $f$  is a  $\gamma_R$ -function,  $w'jj \in B_2$ . Also  $\deg(j) < n-1$ , so there is  $z \in V \setminus \{0, j\}$  such that  $w'jz \notin N(w'jj)$  and  $\deg(w'jz) = \deg(z)$ . Hence,  $f(w'jz) \in \{0, 1\}$ . If  $f(w'jz) = 1$ , then we can move the label 2 from  $w'jj$  to  $w'00$  and the label 1 from  $w'jz$  to  $w'00$ . The function obtained in this manner is a  $\gamma_R$ -function on  $S(G, t)$ , and so we can assume that  $f$  is defined in this manner, i.e.,  $\omega(V_{w0}) = 1$ . Now, If  $f(w'jz) = 0$ , then we



have two possibilities. Either  $f(w'j0) = 2$  or  $f(w'jl) = 2$ , for some  $l \in N(z)$ . The case  $f(w'j0) = 2$  is impossible, as we can put the label 1 to  $w00$  and the label 0 to  $w'jj$ , and the function obtained is a Roman dominating function of weight less than  $f$ , which is a contradiction. Finally, if  $f(w'jl) = 2$ , then we can modify the following weights: we put label 2 to  $w'j0$ , label 0 to  $w'jl$  label 1 to  $w00$ , label 0 to  $w'jj$  and, if  $l \in N(j)$ , then we put label 1 to  $w'lj$ . The function obtained in this manner is a  $\gamma_R$ -function on  $S(G, t)$ , and so we can assume that  $f$  is defined in this manner, *i.e.*,  $\omega(V_{w0}) = 1$ . So  $\sum_{w \in V^{t-2}} \omega(V_{w0}) \geq n^{t-2}$ .

Therefore,  $\gamma_R(S(G, t)) = \omega(f) \geq n^{t-2} + 2n^{t-2}(n-1) = n^{t-2}(2n-1)$ . The proof is completed.  $\square$

Since any graph of order  $n$  having at most one vertex of degree greater than or equal to  $n-2$  is the subgraph of a graph of order  $n$  having exactly one vertex of degree  $n-1$ , Remark 5 and Theorem 8 lead to the following result.

**Theorem 9.** *If  $G$  is a graph of order  $n \geq 4$  having at most one vertex of degree greater than or equal to  $n-2$ , then for any integer  $t \geq 2$ ,  $\gamma_R(S(G, t)) \geq n^{t-2}(2n-1)$ .*

## 4 The particular case of paths

Notice that  $S(P_2, t) \cong P_{2^t}$  and so  $\gamma_R(S(P_2, t)) = \left\lceil \frac{2^{t+1}}{3} \right\rceil$ . From now on we assume that  $n \geq 3$ . Let  $V = \{1, 2, \dots, n\}$  be the vertex set of  $P_n$ , and  $\langle V_{wu} \rangle$  a copy of  $P_n$  in  $S(P_n, t)$  for  $w \in V^{t-2}$  and  $u \in V$ . Set

$$A_{wu} = \begin{cases} \{wui \in V_{wu} : i < u-1\}, & 3 \leq u \leq n; \\ \emptyset, & u = 1, 2. \end{cases}$$

$$B_{wu} = \begin{cases} \{wuj \in V_{wu} : j > u+1\}, & 1 \leq u \leq n-2; \\ \emptyset, & u = n-1, n. \end{cases}$$

Also, let

$$D_i = \left\{ \langle V_{wu} \rangle : \omega(V_{wu}) = \left\lceil \frac{2|A_{wu}|}{3} \right\rceil + \left\lceil \frac{2|B_{wu}|}{3} \right\rceil + i \right\}, \text{ for } i \in \{0, 1\}$$

and

$$D_2 = \left\{ \langle V_{wu} \rangle : \omega(V_{wu}) = \left\lceil \frac{2|A_{wu}|}{3} \right\rceil + \left\lceil \frac{2|B_{wu}|}{3} \right\rceil + j, \text{ for some } j \geq 2 \right\},$$

where the weight  $\omega(V_{wu})$  corresponds to a labelling defined by a  $\gamma_R$ -function on  $S(P_n, t)$ . Also set  $\Lambda = \{\langle V_{wu} \rangle : \deg(wuu) \neq \deg(u) \text{ for } 1 \leq u \leq n\}$ . With these notations in mind we will prove the following Lemmas.

**Lemma 10.** Let  $f = (B_0, B_1, B_2)$  be a  $\gamma_R$ -function on  $S(P_n, t)$ , where  $n \geq 3$ . For any  $w \in V^{t-2}$  and  $u \in V$  there exists  $i \geq 0$  such that  $\langle V_{wu} \rangle \in D_i$ , and  $i \geq 1$  whenever  $V_{wu} \notin \Lambda$ .

*Proof.* Let  $P_r = \langle A_{wu} \rangle$  and  $P_{r'} = \langle B_{wu} \rangle$ . Notice that Theorem 4 leads to  $\gamma_R(\langle A_{wu} \rangle) = \lceil \frac{2r}{3} \rceil$  and  $\gamma_R(\langle B_{wu} \rangle) = \lceil \frac{2r'}{3} \rceil$ . If  $V_{wu} \notin \Lambda$ , then  $\deg(wuu) = \deg(u) \leq 2$ . Since

$$\omega(V_{wu}) = \omega(A_{wu}) + \sum_{wui \notin A_{wu} \cup B_{wu}} f(wui) + \omega(B_{wu}),$$

$\omega(V_{wu}) \geq \omega(A_{wu}) + \omega(B_{wu}) + 1$ . If  $\omega(A_{wu}) \geq \lceil \frac{2r}{3} \rceil$  or  $\omega(B_{wu}) \geq \lceil \frac{2r'}{3} \rceil$ , then we are done. If  $A_{wu} \neq \emptyset$  and  $\omega(A_{wu}) < \lceil \frac{2r}{3} \rceil$ , then  $f(wu(u-2)) = 0$ ,  $f(wu(u-3)) \leq 1$ , and so  $f(wu(u-1)) = 2$ . Hence,  $\omega(A_{wu}) + f(wu(u-1)) = \lceil \frac{2(r-2)}{3} \rceil + 1 + 2 \geq \lceil \frac{2r}{3} \rceil + 1$ . By analogy, if  $B_{wu} \neq \emptyset$  and  $\omega(B_{wu}) < \lceil \frac{2r'}{3} \rceil$ , then  $\omega(B_{wu}) + f(wu(u+1)) \geq \lceil \frac{2r'}{3} \rceil + 1$ . Therefore, in any case,

$$\omega(V_{wu}) \geq \left\lceil \frac{2|A_{wu}|}{3} \right\rceil + \left\lceil \frac{2|B_{wu}|}{3} \right\rceil + 1.$$

Let  $V_{wu} \in \Lambda$ . Then  $wuu \in N(w'vv)$  where  $w' \in V^{t-2}$  and  $v \in V$ . Thus, as above,

$$\omega(V_{wu}) = \omega(A_{wu}) + \sum_{wui \notin A_{wu} \cup B_{wu}} f(wui) + \omega(B_{wu}) \geq \left\lceil \frac{2|A_{wu}|}{3} \right\rceil + \left\lceil \frac{2|B_{wu}|}{3} \right\rceil.$$

□

**Lemma 11.** Let  $V$  be the vertex set of  $P_n$ ,  $n \geq 3$ , and  $t$  a positive integer. If for some  $w \in V^{t-2}$  and  $u \in V$  we have that  $\langle V_{wu} \rangle \in D_0$ , then there exists  $w' \in V^{t-2}$  and  $v \in N_G(u)$  such that  $\langle V_{w'v} \rangle \in D_2$ .

*Proof.* Let  $f = (B_0, B_1, B_2)$  be  $\gamma_R$ -function on  $S(P_n, t)$ , and  $\langle V_{wu} \rangle \in D_0$ . Then  $\sum_{wui \notin A_{wu} \cup B_{wu}} f(wui) = 0$ . Thus,  $wuu \in N(w'vv)$  where  $w'vv \in V^{t-2} \cap B_2$  for  $w' \in V^{t-2}$

and  $v \in V$ . Hence,  $\langle V_{w'v} \rangle \in \Lambda$  and  $\omega(V_{w'v}) \geq \left\lceil \frac{2|A_{w'v}|}{3} \right\rceil + \left\lceil \frac{2|B_{w'v}|}{3} \right\rceil + 2$ . So,  $\langle V_{w'v} \rangle \in D_2$ . □

**Theorem 12.** For any integers  $n \geq 3$  and  $t \geq 2$ ,

$$\gamma_R(S(P_n, t)) = \begin{cases} n^{t-2} (n \lceil \frac{2n}{3} \rceil - \lceil \frac{n}{3} \rceil), & n \equiv 0, 1 \pmod{3}; \\ n^{t-2} (n \lceil \frac{2n}{3} \rceil - 2 \lceil \frac{n}{3} \rceil + 1), & n \equiv 2 \pmod{3}. \end{cases}$$

*Proof.* We first proceed to deduce the lower bound  $\gamma_R(S(P_n, t)) \geq n^{t-2}(n\lceil\frac{2n}{3}\rceil - \lceil\frac{n}{3}\rceil)$ . Let  $V = \{1, 2, \dots, n\}$ , and  $f = (B_0, B_1, B_2)$  a  $\gamma_R$ -function on  $S(P_n, t)$ . Let  $\langle V_{wu} \rangle$  be a copy of  $P_n$  in  $S(P_n, t)$  for  $w \in V^{t-2}$  and  $u \in V$ . Since

$$\gamma_R(S(P_n, t)) = \sum_{w \in V^{t-2}, u \in V} \omega(V_{wu}),$$

we proceed to obtain a lower bounds on  $\omega(V_{wu})$  in terms of  $n$ . Before doing it, notice that

$$\gamma_R(S(P_n, t)) = \sum_{\langle V_{wu} \rangle \in D_0} \omega(V_{wu}) + \sum_{\langle V_{wu} \rangle \in D_1} \omega(V_{wu}) + \sum_{\langle V_{wu} \rangle \in D_2} \omega(V_{wu})$$

and by Lemma 11, there exists an injective application  $\psi : D_0 \rightarrow D_2$ , so that we emphasize that if  $\langle V_{wu} \rangle \in D_0$ , then the contribution of  $\omega(V_{wu}) + \omega(\psi(\langle V_{wu} \rangle))$  to  $\gamma_R(S(P_n, t))$  is greater than or equal to its contribution when both  $\langle V_{wu} \rangle$  and  $\psi(\langle V_{wu} \rangle)$  are in  $D_1$ . With this observation in mind we continue the proof.

By Lemma 10,  $\omega(V_{wu}) = \left\lceil \frac{2|A_{wu}|}{3} \right\rceil + \left\lceil \frac{2|B_{wu}|}{3} \right\rceil + i$ , for some  $i \geq 0$ . Hence, we now proceed to express  $\left\lceil \frac{|A_{wu}|}{3} \right\rceil$  and  $\left\lceil \frac{2|B_{wu}|}{3} \right\rceil$  in terms of  $n$ . To this end, we consider the set  $S = \{x \in V : x \equiv 2 \pmod{3}\}$  and we differentiate three cases.

Case 1:  $n = 3k$  for some positive integer  $k$ . So  $S$  is a  $\gamma$ -set of  $P_n$ . If  $u \in S$ , then  $|A_{wu}|, |B_{wu}| \in \{3k' : 0 \leq k' \leq k-1\}$  and, as  $|A_{wu} \cup B_{wu}| = n-3$ , we have

$$\omega(V_{wu}) = 2\frac{n-3}{3} + i = \frac{2n}{3} + i - 2. \quad (1)$$

If  $u \in N(S) \setminus \{1, n\}$ , then  $|A_{wu}| \in \{l : l \equiv 1 \pmod{3}\}$  and  $|B_{wu}| \in \{l : l \equiv 2 \pmod{3}\}$  or vice versa. Hence,  $\omega(V_{wu}) = \frac{2n}{3} + i - 1$ . Notice that if  $u = 1$ , then  $A_{wu} = \emptyset$  and  $|B_{wu}| \equiv 1 \pmod{3}$ , which implies that  $\omega(V_{wu}) = \frac{2n}{3} + i - 1$ . The case  $u = n$  is analogous to the previous one. Therefore,

$$\begin{aligned} \gamma_R(S(P_n, t)) &= \sum_{w \in V^{t-2}} \sum_{u \in V} \omega(V_{wu}) \\ &\geq n^{t-2} \left( \left( \frac{2n}{3} - 1 \right) \gamma(P_n) + \frac{2n}{3} (n - \gamma(P_n)) \right) \\ &= n^{t-2} \left( n \left\lceil \frac{2n}{3} \right\rceil - \left\lceil \frac{n}{3} \right\rceil \right). \end{aligned}$$

Case 2:  $n = 3k + 1$  for any positive integer  $k$ . In this case,  $S' = S \cup \{n-1\}$  is a  $\gamma$ -set of  $P_n$ . If  $\langle V_{wd} \rangle$  is a copy of  $P_n$  for some  $d \in S'$ ,  $|A_{wd}| \in \{l : l \equiv 0 \pmod{3}\}$  and  $|B_{wd}| \in \{l : l \equiv 1 \pmod{3}\}$  or vice versa. Hence,

$$\omega(V_{wd}) = \left\lceil \frac{2|A_{wd}|}{3} \right\rceil + \left\lceil \frac{2|B_{wd}|}{3} \right\rceil + i = 2 \left\lfloor \frac{n}{3} \right\rfloor + i - 1. \quad (2)$$

Let  $V_{wu}$  where  $u \in N(S') \setminus \{1, n\}$ . Hence, we have two possibilities,  $|A_{wu}|, |B_{wu}| \in \{l : l \equiv 2 \pmod{3}\}$  or  $|A_{wu}|, |B_{wu}| \in \{l : l \equiv 0, 1 \pmod{3}\}$  where  $|A_{wu}| \not\equiv |B_{wu}|$

(mod 3). In the first case,  $\omega(V_{wu}) \geq 2\lfloor \frac{n}{3} \rfloor + i$  and, in the second one,  $\omega(V_{wu}) \geq 2\lfloor \frac{n}{3} \rfloor + i - 1$ .

Suppose that  $\omega(V_{wv}) = 2\lfloor \frac{n}{3} \rfloor + i - 1$  for  $w \in V^{t-2}$  and  $v \in V$ . Then  $\omega(V_{w(v-1)}) > 2\lfloor \frac{n}{3} \rfloor + i - 1$  where  $v - 1 \in S$ . Therefore  $\omega(V_{wu})$  is equal to  $2\lfloor \frac{n}{3} \rfloor + i - 1$  at most for  $\gamma(P_n)$  copies of  $P_n$ , and for other copies it is more than  $2\lfloor \frac{n}{3} \rfloor + i - 1$ . Hence,

$$\gamma_R(S(P_n, t)) \geq n^{t-2} \left( 2\gamma(P_n) \left\lfloor \frac{n}{3} \right\rfloor + (n - \gamma(P_n)) \left( 2 \left\lfloor \frac{n}{3} \right\rfloor + 1 \right) \right) = n^{t-2} \left( n \left\lceil \frac{2n}{3} \right\rceil - \left\lfloor \frac{n}{3} \right\rfloor \right).$$

Case 3:  $n = 3k + 2$  for any positive integer  $k$ . We discuss first words of the form  $wu$  where  $2 \leq u \leq n - 1$  and  $w \in V^{t-2}$ . If  $wuu \in B_2 \cup B_1$ , then  $\omega(V_{wu}) \geq \lceil \frac{2(u-2)}{3} \rceil + \lceil \frac{2(n-u-1)}{3} \rceil + 1$ . Hence  $\omega(V_{wu}) \geq 2\lfloor \frac{n}{3} \rfloor + 1$  for  $u \equiv 0 \pmod{3}$  and  $\omega(V_{wu}) \geq 2\lfloor \frac{n}{3} \rfloor + 2$  for others. Now, suppose that  $wuu \in B_0$  and  $\langle V_{wu} \rangle \notin D_0$ . In this case  $wu(u-1) \in B_2$  or  $wu(u+1) \in B_2$ , say  $wu(u+1) \in B_2$ . Hence,  $\omega(V_{wu}) \geq \lceil \frac{2(u-2)}{3} \rceil + \lceil \frac{2(n-u-2)}{3} \rceil + 2$ , which implies that  $\omega(V_{wu}) \geq 2\lfloor \frac{n}{3} \rfloor + 1$  for  $u \in \{3k', 3k' + 2 : 0 \leq k' \leq k - 1\}$  and  $\omega(V_{w'}) \geq 2\lfloor \frac{n}{3} \rfloor + 2$  for others. In summary,  $\omega(V_{wu}) \geq 2\lfloor \frac{n}{3} \rfloor + 1$  for  $u \equiv 0, 2 \pmod{3}$  and  $\omega(V_{wu}) \geq 2\lfloor \frac{n}{3} \rfloor + 2$  for  $u \equiv 1 \pmod{3}$ .

Now, let  $u \in \{1, n\}$ . Suppose that  $u = 1$  (for  $u = n$ , the proof is likewise). If  $\langle V_{w1} \rangle \in D_2$ , then  $\omega(V_{w1}) \geq 2\lfloor \frac{n}{3} \rfloor + 2$ . Now, if  $\langle V_{w1} \rangle \in D_1$ , then  $f(w11) = 1$  or  $f(w11) = 0$ . In the first case,  $f(w21) = 2$ , as  $f(w13) = 2$  implies that  $\langle V_{w1} \rangle \in D_2$ , which is a contradiction. In the second case, there exists  $w' \in V^{t-2}$  such that  $f(w'22) = 2$  and  $w11 \in N(w'22)$ . As a consequence,  $\omega(V_{w1}) \geq 2\lfloor \frac{n}{3} \rfloor + 2$  or for some  $w' \in V^{t-2}$ ,  $\omega(V_{w'2}) \geq 2\lfloor \frac{n}{3} \rfloor + 2$ . In summary, we can collect the lower bounds for weight of the copies of  $P_n$  in  $S(P_n, 2)$  in a table.

	$u =$	$3k'$	$3k' + 1$	$3k' + 2$
	$u \neq 1, n, \omega(V_{wu}) \geq$	$2\lfloor \frac{n}{3} \rfloor + 1$	$2\lfloor \frac{n}{3} \rfloor + 2$	$2\lfloor \frac{n}{3} \rfloor + 1$
$\langle V_{w1} \rangle \in D_0$	$\omega(V_{w1}) \geq$ $\omega(V_{w2}) \geq$ and $\exists w' \in V^{t-2} : \omega(V_{w'2}) \geq$		$2\lfloor \frac{n}{3} \rfloor$	$2\lfloor \frac{n}{3} \rfloor + 2$  $2\lfloor \frac{n}{3} \rfloor + 2$
$\langle V_{w1} \rangle \in D_1$	$\omega(V_{w1}) \geq$ $\omega(V_{w2}) \geq$ or $\exists w' \in V^{t-2} : \omega(V_{w'2}) \geq$		$2\lfloor \frac{n}{3} \rfloor + 1$	$2\lfloor \frac{n}{3} \rfloor + 2$  $2\lfloor \frac{n}{3} \rfloor + 2$
$\langle V_{w1} \rangle \in D_2$	$\omega(V_{w1}) \geq$		$2\lfloor \frac{n}{3} \rfloor + 2$	

Therefore,

$$\begin{aligned} \gamma_R(S(P_n, t)) &= \sum_{w \in V^{t-2}} \sum_{u \in V} \omega(V_{wu}) \\ &\geq n^{t-2} \left( \left( 2 \left\lfloor \frac{n}{3} \right\rfloor + 1 \right) \left( 2 \left\lfloor \frac{n}{3} \right\rfloor + 1 \right) + \left\lceil \frac{n}{3} \right\rceil \left( 2 \left\lfloor \frac{n}{3} \right\rfloor + 2 \right) \right) \\ &= n^{t-2} \left( n \left\lceil \frac{2n}{3} \right\rceil - 2 \left\lfloor \frac{n}{3} \right\rfloor + 1 \right). \end{aligned}$$

and the proof of the lower bound is complete.

For  $n \equiv 0, 1 \pmod{3}$ , the upper bound  $\gamma_R(S(P_n, t)) \leq n^{t-2}(n\lceil\frac{2n}{3}\rceil - \lceil\frac{n}{3}\rceil)$  is obtained from Theorem 6. Thus we consider the case  $n = 3k + 2$  for some positive integer  $k$ .

As above, consider the set  $S = \{x \in V \setminus \{n\} : x \equiv 2 \pmod{3}\}$ . In order to construct a Roman dominating function we introduce the following sets.

$$\begin{aligned} A_1 &= \{wis : w \in V^{t-2}, s \in S, i \geq s + 2\}, \\ A_2 &= \{wi(n-1) : w \in V^{t-2}, i \in \{1, n\}\}, \\ A_3 &= \{wij : w \in V^{t-2}, 1 \leq i \leq n-2, j = i+1+3k', 0 \leq k' \leq k-1\}, \\ C_1 &= \{win : w \in V^{t-2}, i \in S\}, \\ C_2 &= \{w(s+1)(s-1) : w \in V^{t-2}, s \in S\}, \\ C_3 &= \{w(n-1)(n-1) : w \in V^{t-2}\}. \end{aligned}$$

Define  $g : V^t \rightarrow \{0, 1, 2\}$  such that

$$g(wij) = \begin{cases} 2, & wij \in \bigcup_{i=1}^3 A_i; \\ 1, & wij \in \bigcup_{i=1}^3 C_i; \\ 0, & \text{otherwise.} \end{cases}$$

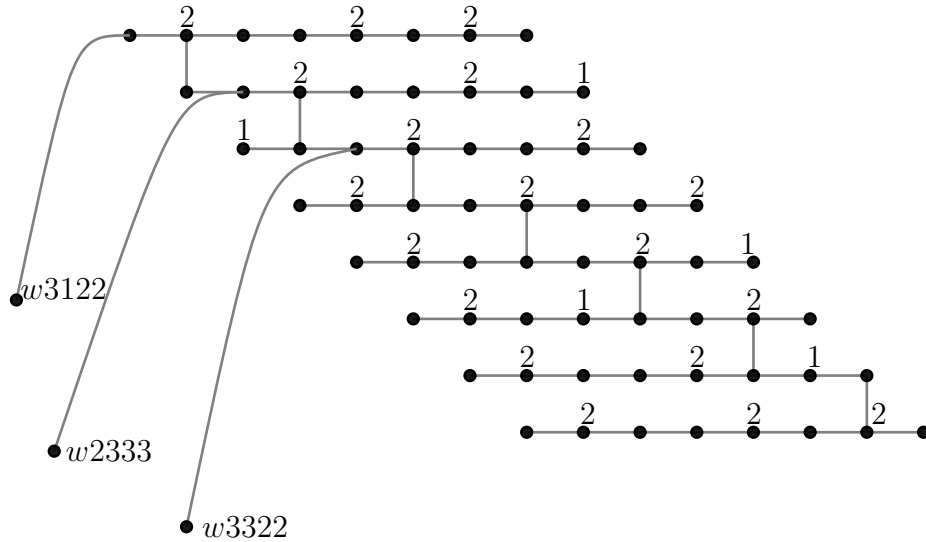


Figure 4: This figure shows the labelling of  $\langle V_{w32} \rangle \cong S(P_8, 2)$  induced by  $g$ , where labels 0's are omitted.

Suppose that  $g(wij) = 0$  for  $w \in V^{t-2}$  and  $i, j \in V$ . If  $i > j + 2$ , then  $j \notin S$  and so  $wij \in N(wis)$  where  $s \in \{j-1, j+1\}$ . As consequence,  $i > s + 2$  and  $wis \in A_1$ .

If  $i = j + 2$  and  $s = j + 1$ , then  $wij = w(s + 1)(s - 1) \in C_2$ , which is a contradiction, as  $g(wij) = 0$ . Hence, if  $i = j + 2$ , then  $s = j - 1$  and  $wis = w(s + 3)s \in A_1$ . Now, let  $i < j + 2$ . If  $i = j + 1$ , then  $wij = wi(i - 1) \in N(w(i - 1)i)$  and  $w(i - 1)i \in A_2$ . Also, if  $i < j + 1$ , then  $wij$  is dominated by some vertex in  $A_2 \cup A_3$ . Hence,  $g$  is a Roman dominating function on  $S(P_n, t)$ . Thus,

$$\gamma_R(S(P_n, t)) \leq \omega(g) = 2 \sum_{i=1}^3 |A_i| + \sum_{i=1}^3 |C_i|.$$

On one hand,

$$\sum_{i=1}^3 |C_i| = n^{t-2}(2|S| + 1) = n^{t-2}(2\gamma(P_n) - 1) = 2k + 1$$

and, on the other hand,

$$|A_1| = n^{t-2} \sum_{u \in S} u = n^{t-2} \left( \frac{3k^2 + k}{2} \right), \quad |A_2| = 2n^{t-2}$$

and

$$|A_3| = n^{t-2} \left( k + 2 + \sum_{i=2}^k 3i \right) = n^{t-2} \left( \frac{3k^2 + 5k - 2}{2} \right).$$

Thus,

$$\sum_{i=1}^3 |A_i| = n^{t-2} \left( \frac{3k^2 + k}{2} + 2 + \frac{3k^2 + 5k - 2}{2} \right) = n^{t-2}(3k^2 + 3k + 1).$$

Therefore,  $\gamma_R(S(P_n, t)) \leq n^{t-2}(6k^2 + 8k + 3)$  and, since  $n = 3k + 2$ ,

$$\gamma_R(S(P_n, t)) \leq n^{t-2} \left( n \left\lceil \frac{2n}{3} \right\rceil - 2 \left\lceil \frac{n}{3} \right\rceil + 1 \right),$$

as required. □

## 5 The particular case of cycles

**Theorem 13.** *Let  $n \geq 4$  and  $t \geq 2$  be two integers. If  $n \equiv 1, 2 \pmod{3}$ , then  $\gamma_R(S(C_n, t)) = n^{t-1} \lfloor \frac{2n}{3} \rfloor$ , otherwise,  $\frac{n^{t-1}(2n-3)}{3} \leq \gamma_R(S(C_n, t)) \leq \frac{n^{t-1}(2n-1)}{3}$ .*

*Proof.* Let  $V = \{1, \dots, n\}$  be the vertex set of  $C_n$ , where  $i \in N(i + 1)$ , for any  $i$ , and the addition is taken modulo  $n$ . First, we proceed to deduce the upper bound for  $\gamma_R(S(C_n, t))$ . If  $n \equiv 0 \pmod{3}$ , then Theorem 6 leads to

$$\gamma_R(S(C_n, t)) \leq \frac{n^{t-1}}{3}(2n - 1). \quad (3)$$

Suppose that  $n = 3k+1$ , for some integer  $k$ . Define  $D = \{ij : i \in V, j = i+1+3k', 0 \leq k' < k-1\}$  and  $D_{t-2} = \{wx : w \in V^{t-2}, x \in D\}$ . Notice that  $D$  is a 2-packing dominating set, and  $D \cap \{ii : i \in V\} = \emptyset$ , hence  $D_{t-2}$  is also a 2-packing dominating set and therefore  $\gamma(S(C_n, t)) = |D_{t-2}| = n^{t-2}|D| = n^{t-1} \left\lfloor \frac{n}{3} \right\rfloor$ , which implies that

$$\gamma_R(S(C_n, t)) \leq 2\gamma(S(C_n, t)) = n^{t-1} \left\lfloor \frac{2n}{3} \right\rfloor. \quad (4)$$

Now, let  $n = 3k + 2$  for any positive integer  $k$ . Set

$$A = \{wij : w \in V^{t-2}, i \in V, j = i + 1 + 3k', 0 \leq k' \leq k-1\}$$

and

$$B = \{wij : w \in V^{t-2}, i \in V, j = i - 2\}.$$

Define  $f_2 : V^t \rightarrow \{0, 1, 2\}$  such that

$$f_2(x) = \begin{cases} 2, & x \in A; \\ 1, & x \in B; \\ 0, & \text{otherwise.} \end{cases}$$

Let  $w \in V^{t-2}$  and  $i, j \in V$  such that  $g(wii) = 0$ . If  $j \equiv i - 1 \pmod{n}$ , then  $wi(i-1) \in N(w(i-1)i) \subset N(A)$ . Otherwise,  $j \equiv i + 3k'$  or  $i + 2 + 3k' \pmod{n}$ , for  $1 \leq k' \leq k-1$ . Hence,  $wij \in N(wi(j+1))$  or  $wij \in N(wi(j-1))$  respectively. So  $wij \in N(A)$ . Therefore,  $g$  is a Roman dominating function on  $S(C_n, t)$  and, as a consequence,

$$\gamma_R(S(C_n, t)) \leq \omega(f_2) = 2|A| + |B| = n^{t-1}(2k+1) = n^{t-1} \left\lfloor \frac{2n}{3} \right\rfloor. \quad (5)$$

Now we will find the lower bound for  $\gamma_R(S(C_n, t))$ . Assume that  $f = (B_0, B_1, B_2)$  is a  $\gamma_R$ -function on  $S(C_n, t)$ . Set

$$C_{wu} = \{wui \in V_{wu} : i \notin \{u-1, u, u+1\}\}$$

for  $w \in V^{t-2}$  and  $u \in V$ . Hence, induced subgraph on  $C_{wu}$  is isomorphic to  $P_{n-3}$  and

$$\omega(V_{wu}) = \omega(C_{wu}) + \sum_{i \in \{u-1, u, u+1\}} f(wui). \text{ Let}$$

$$D_i = \left\{ \langle V_{wu} \rangle : \omega(V_{wu}) = \left\lfloor \frac{2n}{3} \right\rfloor - 2 + i \right\} \text{ for } i \in \{0, 1\}$$

and

$$D_2 = \left\{ \langle V_{wu} \rangle : \omega(V_{wu}) = \left\lfloor \frac{2n}{3} \right\rfloor - 2 + j, \text{ for some } j \geq 2 \right\}.$$

Notice that

$$\gamma_R(S(C_n, t)) = \sum_{\langle V_{wu} \rangle \in D_0} \omega(V_{wu}) + \sum_{\langle V_{wu} \rangle \in D_1} \omega(V_{wu}) + \sum_{\langle V_{wu} \rangle \in D_2} \omega(V_{wu}).$$

If  $\langle V_{wu} \rangle \in D_0$ , then  $\{wu(u-1), wuu, wu(u+1)\} \subset B_0$  and so there exists  $w' \in V^{t-2}$  and  $v \in V$  such that  $wuu \in N(w'vv)$  and  $f(w'vv) = 2$ . Thus,  $\langle V_{w'v} \rangle \in D_2$ . We can define an injective application  $\phi : D_0 \rightarrow D_2$ , so that we emphasize that if  $\langle V_{wu} \rangle \in D_0$ , then the contribution of  $\omega(V_{wu}) + \omega(\phi(\langle V_{wu} \rangle))$  to  $\gamma_R(S(C_n, t))$  is greater than or equal to such contribution when both  $\langle V_{wu} \rangle$  and  $\phi(\langle V_{wu} \rangle)$  are in  $D_1$ . The argument shows that,

$$\gamma_R(S(C_n, t)) = \sum_{w \in V^{t-2}} \sum_{u \in V} \omega(V_{wu}) \geq n^{t-1} \left( \left\lceil \frac{2n}{3} \right\rceil - 1 \right).$$

Therefore, the result follows.  $\square$

## 6 The particular case of complete graphs

The domination number of  $S(K_n, t)$  was previously studied by Klavžar, Milutinović and Petr in [13] where they obtained the following result.

**Theorem 14.** [13] *For any integers  $n \geq 2$  and  $t \geq 1$ ,*

$$\gamma(S(K_n, t)) = \begin{cases} \frac{n^t + n}{n + 1}, & t \text{ even}; \\ \frac{n^t + 1}{n + 1}, & t \text{ odd}. \end{cases}$$

The above result is an important tool to deduce an upper bound on the Roman domination number of  $S(K_n, t)$ .

**Theorem 15.** *For any integers  $n \geq 2$  and  $t \geq 1$ ,*

$$\gamma_R(S(K_n, t)) \leq \begin{cases} \frac{2n^t + n - 1}{n + 1}, & t \text{ even}; \\ \frac{2(n^t + 1)}{n + 1}, & t \text{ odd}. \end{cases}$$

*Proof.* Let  $V = \{1, 2, \dots, n\}$  be the vertex set of  $K_n$ . For  $t$  odd we deduce the bound from Theorem 14, as  $\gamma_R(S(K_n, t)) \leq 2\gamma(S(K_n, t))$ . We claim that for  $t = 2k$  there exists a Roman dominating function such that  $f(1 \dots 1) = 1$  and  $\omega(f) = \frac{2n^{2k} + n - 1}{n + 1}$ . To show this we proceed by induction on  $k$ . For  $k = 1$  we define the Roman dominating function  $f$  as follows.  $f(11) = 1$ ,  $f(i1) = 2$  for all  $i \neq 1$  and  $f(xy) = 0$  for others. Notice that  $\omega(f) = 2(n - 1) + 1 = \frac{2n^2 + n - 1}{n + 1}$ .

Now, suppose that  $f$  is a Roman dominating function on  $S(K_n, 2k)$  such that  $f(1 \dots 1) = 1$  and  $\omega(f) = \frac{2n^{2k} + n - 1}{n + 1}$ . We shall construct a Roman dominating function  $f'$  on  $S(K_n, 2k + 2)$  in the following way:

- $f'(11w) = f(w)$  for all  $w \in V^{2k}$ .



- $f'(1i \dots i) = 0$  for all  $i \neq 1$  and  $f(11w) = f(w')$  for all  $w \in V^{2k-2} \setminus \{i \dots i : i \in V\}$ , where  $w'$  is obtained from  $w$  by exchanging  $i$  and 1.
- For any  $i \in V \setminus \{1\}$  and  $w \in V^{2k}$ , we define  $f(i1w)$  as follows. As shown in [13, Corollary 3.5], there exists a 1-perfect code  $C$  of  $S(K_n, 2k)$  which contains all the extreme vertices. So, we set  $f'(i1w) = 2$  for all  $w \in C$  and  $f'(i1w) = 0$  for others.
- $f'(ij1 \dots 1) = 0$  and  $f'(ijw) = f(w)$  for all  $i, j \neq 1$  and  $w \neq 1 \dots 1$ .

Notice that  $f'(1 \dots 1) = 1$ . To conclude that  $f'$  is a Roman dominating function on  $S(K_n, 2k+2)$  we only need to observe that all  $x \in V^{2k+2}$  of the form  $x = 1i \dots i$ ,  $i \neq 1$  are adjacent to  $i1 \dots 1$  and  $f'(i1 \dots 1) = 2$ , and all  $x \in V^{2k+2}$  of the form  $x = ij1 \dots 1$ ,  $i, j \neq 1$  are adjacent to  $i1j \dots j$  and  $f'(i1j \dots j) = 2$ . Finally, by Theorem 14,  $|C| = \frac{n^{2k+n}}{n+1}$ , and so

$$\omega(f') = \omega(f) + (n-1)(\omega(f) - 1) + 2|C|(n-1) + (n-1)^2(\omega(f) - 1) = \frac{2n^{2k+2} + n - 1}{n + 1},$$

as required.  $\square$

By Remark 5 we deduce the following corollary.

**Corollary 16.** *For any graph  $G$  of order  $n$  and any integer  $t$ ,*

$$\gamma_R(S(G, t)) \geq \gamma_R(S(K_n, t)).$$

As the above corollary shows, a lower bound (or a closed formula) on the Roman domination number of  $S(K_n, t)$  imposes a lower bound on  $\gamma_R(S(G, t))$  for every graph  $G$ . Therefore, this issue definitely deserves further research.

## References

- [1] S. Bermudo, H. Fernau, J. M. Sigarreta, The differential and the Roman domination number of a graph, *Applicable Analysis and Discrete Mathematics* 8 (1) (2014) 155–171.
- [2] M. Chellali, T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi, A. A. McRae, A roman domination chain, *Graphs and Combinatorics* 32 (1) (2016) 79–92.
- [3] E. J. Cockayne, P. A. Dreyer Jr., S. M. Hedetniemi, S. T. Hedetniemi, Roman domination in graphs, *Discrete Mathematics* 278 (1–3) (2004) 11–22.
- [4] A. Estrada-Moreno, E. D. Rodríguez-Bazan, J. A. Rodríguez-Velázquez, On the General Randić index of polymeric networks modelled by generalized Sierpiński graphs, *arXiv:1510.07982 [math.CO]*.
- [5] J. Geetha, K. Somasundaram, Total coloring of generalized Sierpiński graphs, *The Australasian Journal of Combinatorics* 63 (2015) 58–69.

- [6] S. Gravier, M. Kovše, M. Mollard, J. Moncel, A. Parreau, New results on variants of covering codes in Sierpiński graphs, *Designs, Codes and Cryptography* 69 (2) (2013) 181–188.
- [7] S. Gravier, M. Kovše, A. Parreau, Generalized Sierpiński graphs, in: *Posters at EuroComb'11*, Rényi Institute, Budapest, 2011.  
URL <http://www.renyi.hu/conferences/ec11/posters/parreau.pdf>
- [8] M. A. Henning, Defending the roman empire from multiple attacks, *Discrete Mathematics* 271 (1–3) (2003) 101–115.
- [9] A. M. Hinz, C. H. auf der Heide, An efficient algorithm to determine all shortest paths in Sierpiński graphs, *Discrete Applied Mathematics* 177 (0) (2014) 111 – 120.
- [10] A. M. Hinz, S. Klavžar, U. Milutinović, C. Petr, *The Tower of Hanoi - Myths and Maths*, Birkhäuser/Springer Basel, 2013.
- [11] A. M. Hinz, S. Klavžar, S. S. Zemljich, A survey and classification of Sierpiński-type graphs, Submitted.
- [12] S. Klavžar, U. Milutinović, Graphs  $S(n, k)$  and a variant of the Tower of Hanoi problem, *Czechoslovak Mathematical Journal* 47 (1) (1997) 95–104.
- [13] S. Klavžar, U. Milutinović, C. Petr, 1-Perfect codes in Sierpiński graphs, *Bulletin of the Australian Mathematical Society* 66 (3) (2002) 369–384.
- [14] S. Klavžar, I. Peterin, S. S. Zemljich, Hamming dimension of a graph — The case of Sierpiński graphs, *European Journal of Combinatorics* 34 (2) (2013) 460 – 473.
- [15] S. Klavžar, S. S. Zemljich, On distances in Sierpiński graphs: Almost-extreme vertices and metric dimension, *Applicable Analysis and Discrete Mathematics* 7 (1) (2013) 72–82.
- [16] J. A. Rodríguez-Velázquez, E. Estaji, The strong metric dimension of generalized Sierpiński graphs with pendant vertices, *Ars Mathematica Contemporanea*, to appear.
- [17] J. A. Rodríguez-Velázquez, E. D. Rodríguez-Bazan, A. Estrada-Moreno, On Generalized Sierpiński Graphs, *Discussiones Mathematicae Graph Theory*, to appear.
- [18] J. A. Rodríguez-Velázquez, J. Tomás-Andreu, On the Randić Index of Polymer Networks Modelled by Generalized Sierpiński Graphs, *MATCH Communications in Mathematical and in Computer Chemistry* 74 (1) (2015) 145–160.
- [19] F. Xueliang, B. Y. Yuansheng, J. Baoqia, Roman domination in regular graphs, *Discrete Mathematics* 309 (2009) 1528–153.

- [20] I. G. Yero, J. A. Rodríguez-Velázquez, Roman domination in Cartesian product graphs and strong product graphs, *Applicable Analysis and Discrete Mathematics* 7 (2) (2013) 262–274.